Travelling to exotic places with ultracold atoms

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Abstract. We review and speculate on two recent developments of quantum optics and ultracold atoms. First, we discuss a possible realization of “phonon” physics, or as we call it *refracton* physics with optical lattices. To this aim we combine the physics of cold atoms with cavity QED, and investigate superfluid – Mott insulator quantum phase transition. The systems can exhibit cavity mode modifications due to local changes of refraction index (*refractons*). Second, we discuss the physics of strongly correlated particles in Abelian, and more interestingly in non-Abelian magnetic fields, using cold atoms.

INTRODUCTION

This paper has been presented as the final invited lecture at the International Conference on Atomic Physics, ICAP 2006, held in Innsbruck in July 2006. The speaker, M. Lewenstein has met all the advantages and disadvantages of being the last. The audience began to shrink and people were definitely starting to be tired of atomic physics, despite the enormous success of the ICAP 2006. On the other hand, being the last speaker M. Lewenstein had the opportunity to speculate in his lecture upon the future of atomic physics, and the physics of cold atoms in particular, and to thank truly the organisers.

This paper consist of two relatively independent parts. In the first one we surmise the possibility of realising fascinating phenomena resembling those occurring in solid state physics due to the presence of phonons. We start by reviewing the superfluid-Mott insulator (SF–MI) transition of bosonic atoms in an optical lattice, which is a paradigm of a quantum phase transition. We turn then to the SF–MI transition inside an optical cavity. In the simplest situation, the atoms influence the cavity field by shifting the cavity resonance. The resulting effective Bose-Hubbard model suggest the existence of overlapping, competing Mott phases for wide range of parameters, and bistable behavior in the vicinity of the shifted cavity resonance. In a more sophisticated version, the model includes the possibility of *local modifications* of the cavity field due to the presence of atoms and local change of refractive index i.e. *refracton physics*.

The second part deals with ultracold atomic gases in "artificial" non-Abelian fields. Again we review first distinct ways of realizing Abelian and non-Abelian magnetic field

in optical lattices using various types of laser manipulations etc. We present our recent results dealing with the problem of a single atom in a lattice, which, in the presence of Abelian (non-Abelian) magnetic fields, leads to a spectrum in the form of the famous Hofstadter butterfly (or yet not so famous *moth*). Finally, we speculate on non-Abelian fractional Hall effect with ultracold atoms.

TOWARD "REFRACTON" PHYSICS

Introduction. Ultracold atomic gases in optical lattices (OL) are nowadays a subject of very intensive studies, since they provide an unprecedented possibility to study numerous challenges of quantum many body physics (for reviews see [1, 2]). In particular, such systems allow to realize various versions of Hubbard models [3], a paradigm of which is the Bose-Hubbard model [4]. This model exhibits supefluid (SF) – Mott insulator (MI) quantum phase transition [5], and its atomic realization has been proposed in the seminal Ref. [6], followed by the seminal experiments of Ref. [7]. Several aspect and modifications of SF – MI quantum phase transition, or better to say crossover [8], has been intensively studied recently (cf. [2, 9]). The control parameter of the SF – MI transition is the strength of the optical potentials, i.e. laser intensity. For low laser intensities, bosonic atoms can easily tunnel around the lattice, and they condense into a maximally delocalized states with long range phase coherence, and density (on–site atom number) fluctuations. When laser intensity increases, optical potential barriers prevent tunneling. In effect, density fluctuations become costly, and the systems enters a gaped, incompressible MI phase, with no phase coherence, and reduced density fluctuations.

The critique of pure perfectionism. Optical lattices provide an ideal toolkit for Hubbard models [3], since they are robust and do not exhibit phonons – in this sense they are *perfect*. As we often encounter in everyday life, perfectionism has positive and negative sides. In particular, the absence of phonons limits the whole richness of condensed matter phenomena that result from electron-phonon interactions, such as for instance the famous Peierls instability. Antiferromagnetic (non-frustrated) Heisenberg spin systems in perfect lattices, and in particular spin chains in 1D at zero temperature often attain Neél ordering. In the presence of phonons, the system arranges in such a way that every second site move slightly to left, and every other second to the right, so that the energy of spin interactions is modulated: strong-weak-strong-.... The ground state has a form of a chain of dimers (singlets) located at the "strong" bonds. Contrary to Neél antiferromagnets which magnetize in arbitrarily small magnetic fields H, a dimerized antiferromagnet requires a a non-zero magnetization M to break the dimers. Antiferromagnetic spin chains exhibit thus a magnetization plateau at $M = 0$. Magnetization plateaux with much complex origin have been recently intensively discussed in the literature (cf. [10] and references therein). For instance, a simple XXZ spin chain, in an adiabatic approximation for "heavy" phonons is described by the Hamiltonian:

$$H = \frac{1}{2}\sum_i \delta_i^2 + J\sum_i (1 - A\delta_i)(s_i^\dagger s_{i+1} + s_{i+1}^\dagger s_i) + \Delta \sum_i s_{z,i} s_{z,i+1}, \tag{1}$$

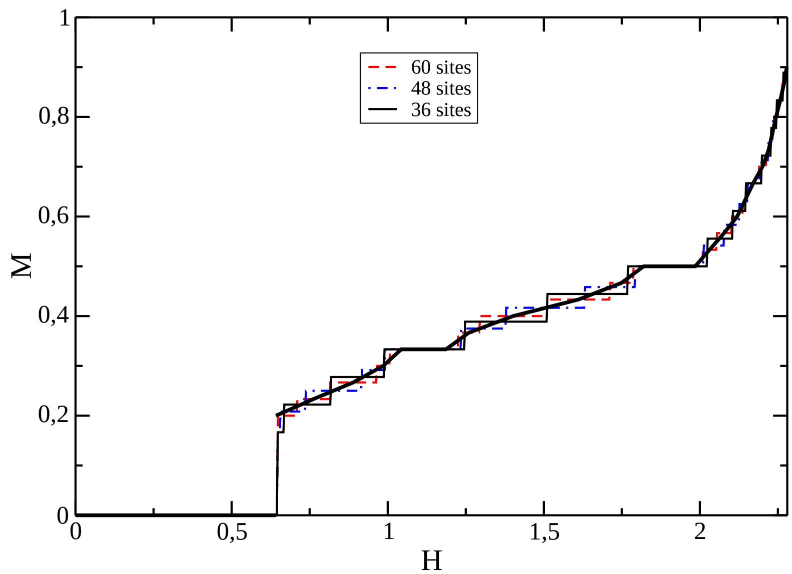

Figure 1. Magnetization plateaux in a frustrated Heisenberg spin chain analogous to the one described by expression (1) (from [10]).

where δ_i are lattice sites displacements, J and Δ nearest neigbour XY and Ising spin couplings respectively, and A describes the XY coupling modifications due to site displacements. This system is expected to exhibit novel magnetization plateaux at non-trivial rational values $M = 1/3, 1/2..$ (see Fig. 1). In the ground state, the displacement attains the self-consistent value $\delta_i = \langle s_i^\dagger s_{i+1} + s_{i+1}^\dagger s_i \rangle$. It is challenging and fascinating to ask whether such physics can be captured and mimicked by cold atoms in lattices that are not perfect and react to their presence. A natural candidate for the realization of such systems is offered by Cavity Quantum Electrodynamics (CQED) [11].

Cavity QED and cold atoms. In fact, in CQED the atoms interact with the cavity mode and actively affect the cavity induced optical lattice. Several studies address the scaling of cavity QED dynamics with the number of atoms and their temperature. Cavity QED techniques were used to measure pair correlations in atom laser [12]. Self-organization of atoms in longitudinally and transversally pumped cavity was observed in [13], and theoretically characterized in [14]. Bragg scattering of atomic structures, which built up in the potential of the optical resonator, has been investigated in [15]. The interactions and cooling of a single atom in a quantum optical lattice in a cavity was studied in Refs. [16, 17]. Maschler and Ritsch [18] implemented the Bose-Hubbard model for atoms in the one-dimensional potential of an optical resonator, which was longitudinally pumped. They showed that the coefficients determining the dynamics, depend on the number of atoms and exhibit long range interactions, due to the photon-mediated forces. In a recent work, we have applied the model developed in [18] to study how the quantum phases of ultracold atomic gases are modified by the photon-mediated long-range interaction due to the optical resonator [19]. We focused onto the superfluid - Mott insulator phase transition for gases of few hundred atoms, and apply

the methods of quantum statistical physics, namely strong coupling expansion [20] to calculate the boundaries of the Mott insulating phases in the $\mu-\eta$ plane, where μ is the chemical potential, while η characterizes the pump strength. We predicted existence of overlapping, competing Mott phases, that may even consist of two not connected regions for a wide range of parameters. We predicted also dispersive bistable behavior [21] in the vicinity of the shifted cavity resonance in the strong coupling regime.

Our model. Our model considers atoms confined in one dimension inside an optical resonator, which is pumped by a classical field. The atomic dipole transition is far-off resonance from the cavity mode, which induces a dipole potential on the atomic ground state. Using the notation of [18], the single-particle Hamiltonian reads:

$$H_0 = \frac{p^2}{2m} + \hbar U_0 \cos^2(kx) n_{\mathrm{ph}} - \hbar\Delta_c n_{\mathrm{ph}} - i\hbar\eta \left(a - a^\dagger\right). \tag{2}$$

Here, p and m are the atomic momentum and mass, η is the amplitude of the pump at frequency ω_p, $\Delta_a = \omega_p - \omega_a$ and $\Delta_c = \omega_p - \omega_c$ are the detunings of the pump from atom and cavity frequencies, $g(x) = g_0\cos(kx)$ is the atom-cavity mode coupling, $k = \omega_c/c$ the mode wave vector, $a^\dagger$ and a the creation and annihilation operators of a photon $\hbar\omega_c$, $n_{\mathrm{ph}} = a^\dagger a$ the number of photon, and $U_0 = g_0^2/\Delta_a$ is the depth of the single-photon dipole potential. The many-body Hamiltonian is obtained from Eq. (2) in second quantization with the atomic field operators $\Psi(x)$, and including the atomic contact interaction [18]. In the bad-cavity limit we eliminate, from the atomic dynamics, the cavity field variables and the number of photons takes the value

$$n_{\mathrm{ph}} \approx \frac{\eta^2}{\kappa^2 + \left[\Delta_c - U_0 \int \mathrm{d}x \cos^2(kx)\Psi^\dagger(x)\Psi(x)\right]^2}, \tag{3}$$

with κ the rate of cavity damping. The cavity potential depends thus non-linearly on the atomic density. In particular, certain atomic phases correspond to resonances, which increase the number of photons and thus the optical lattice depth. This is the salient physical property of this system, which gives rise to novel dynamics, as we will show. We now consider the regime of validity of the tight binding approximation (TBA) and expand the atomic field operators in the lowest energy band as $\Psi(x) = \sum_i b_i w(x-x_i)$, where b_i is the atomic annihilation operator at site i, and $w(x-x_i)$ is the Wannier state localized around site i, which depends on the photon number n_{ph}, and thus on the atomic density. In the Wannier expansion we keep on-site and nearest-neighbour couplings, and neglect all other couplings, and obtain the Hamiltonian

$$H/U = -tB + \frac{1}{2}\sum_i n_i(n_i-1) - gB^2 - \mu N, \tag{4}$$

where $N = \sum_i n_i = \sum_i b_i^\dagger b_i$ is the atom number operator, $B = \sum_i b_i^\dagger b_{i-1} + \mathrm{h.c.}$ is the hopping term, and U denotes the strength of on–site interactions. The parameters

$$t = -\frac{E_1}{U} + \frac{\eta^2\hbar U_0 J_1\left(\kappa^2-\zeta^2\right)}{U\left(\kappa^2+\zeta^2\right)^2};\ \ g = -\frac{\eta^2\hbar U_0^2 J_1^2 \zeta\left(3\kappa^2-\zeta^2\right)}{U\left(\kappa^2+\zeta^2\right)^3}, \tag{5}$$

denote tunneling and long-range coupling respectively, and are expressed in terms of the N-dependent term $\zeta = \Delta_c - U_0 J_0 N$ and of the integrals $E_\ell = \int dx\, w(x - x_l)(-\hbar^2/2m)(d^2/dx^2)w(x - x_{l+\ell})$ and $J_\ell = \int dx\, w(x - x_l)\cos^2(kx)w(x - x_{l+\ell})$, with $\ell = 0, 1$, where we have assumed $J_1 \ll J_0$. Note that in Eq. (4) the number of particles is conserved, since $[N, H] = 0$.

SF - MI transition in a cavity. In [19] we have generalized and applied the method of H. Monien's group [20] (for more references see [2]) to calculate the lobes for a modified 1D Bose-Hubbard model (4) describing atoms in an optical lattice created by pumping a laser beam into the cavity. Technically, this corresponds of using perturbation theory in B to calculate the ground state energy, and the energies of excited states with one additional atom (one hole) present and comparing them. The major difference to the standard case is that the intensity of the cavity field depends on the number of atoms present, since the atoms shift collectively the cavity resonance. Thus the coefficients t, U, g become very complicated functions of the cavity detuning, intensity of the pumping laser, N, etc. Moreover, quantum fluctuations of the resonance shift induce long range interactions between the atoms, proportional to g and hence the parameters t, U, g have to be calculated self-consistently. Moreover these calculations sometimes exhibit bistability effect! Instead of using μ and t to describe the phase diagram, it is necessary to use a proper control parameter which is the strength of the pumping laser. A typical phase diagram as a function of μ in the recoil units, and κ/η is shown in Fig. 2. The striking effect is the overlap a different Mott phases, and even a presence disconnected Mott regions, which follows from the fact that the expressions for t and U for $n = 1, 2, 3, \ldots$ Mott phases are different.

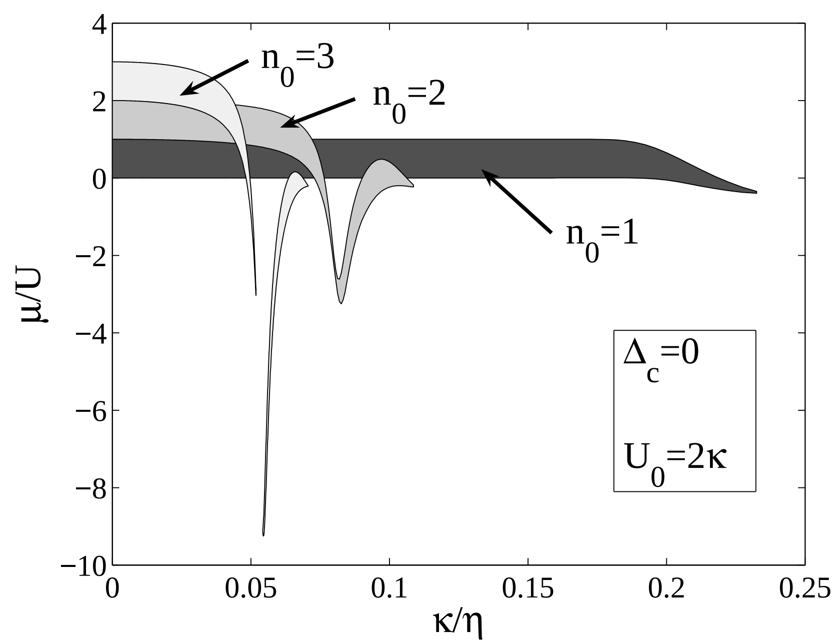

Figure 2. Overlapping and disconnected Mott insulator regions for $\Delta_c = 0$ and $U_0 = 2\kappa$.

Finally, if Δ_a and Δ_c have the same sign, and one may encounter resonances such that $\Delta_c - U_0 J_0 N = 0$; the system may exhibit a bistable behaviour, as reported in [19]

Toward genuine "refractons". An obvious challenging question is whether one can generalize the above results to a situation when the atoms affect the cavity mode *locally*. Such a regime of CQED is experimentally feasible and has been recently considered by D. Meiser and P. Meystre [22], who termed it "the Super-Strong Coupling Regime of Cavity Quantum Electrodynamics". We present here a somewhat alternative approach based on the techniques that one of us developed together with R. Glauber, T. Mossberg and K. Zakrzewski to describe spontaneous emission and Lamb shift modifications in dielectric media and/or in "colored" photonic reservoirs [23]. The idea is to use the full quantum electrodynamical description of the electric field inside the cavity, i.e. expand it into the continuum of modes that have a significant maximum in the density of modes at the cavity resonance. The single atom Hamiltonian becomes then

$$\begin{aligned} H_0 &= \frac{p^2}{2m} + \hbar U_0 \hat{E}^\dagger(x)\hat{E}(x) + \int dk\, ck\, \hat{a}^\dagger(k)\hat{a}(k) \\ &- i\hbar\eta \int dk\, p(k) \left(\hat{a}(k)\exp\left(-ic|k|t\right) - \hat{a}^\dagger(k)\exp\left(+ic|k|t\right)\right)\cos\left(kx\right), \end{aligned} \quad (6)$$

where $\hat{a}(k)$, $\hat{a}^\dagger(k)$ are photon annihilation and creation operators of photons in standing wave modes (that inside the cavity may be well described by $\cos(kx)$). The pumping laser pulse is expanded into these modes, and the amplitude of expansion, $p(k)$ is strongly peaked at $k \simeq k_L$, and $\omega = c|k| \simeq c|k_L| = \omega_L$. The electric field operator $\hat{E}(x)$ inside the cavity can be expanded as (for simplicity we consider only even modes):

$$\hat{E}(x) \propto \int dk \frac{\kappa}{(k - \omega_c + i\kappa)} \hat{a}(k)\cos(kx) = \frac{\eta\cos(k_L x)}{\kappa - i(\Delta_c - U_0\hat{\delta}(x))}, \quad (7)$$

where the final expression appears after adiabatic elimination of the cavity field. The AC Stark shift (which was given by $\delta = J_0\hat{N} + J_1\hat{B}$) becomes now *local*, i.e. x_i-dependent:

$$\hat{\delta}(x_i) = J_0 \sum_j 1/2\left(\exp(-\kappa|x_i - x_j|) + \exp(-\kappa|x_i + x_j|)\right)\hat{n}_j + J_1\hat{B} \quad (8)$$

with $\hat{n}_j$ being the atom number operator at the site j.

In the hard boson limit we end up with the Hamiltonian:

$$\begin{aligned} H = &- \sum_i \left[J(i,\{s_{z,j}\})s_i^\dagger s_{i+1} + s_{i+1}^\dagger s_i J(i,\{s_{z,j}\})\right] \\ &- \sum_i \left[\Delta(i,\{s_{z,j}\})s_i^\dagger s_i + s_i^\dagger s_i \Delta(i\{s_{z,j}\})\right] - \sum_i H s_{z,i}, \end{aligned} \quad (9)$$

Apart from the fact that the system is ferromagnetic, it resembles to great extend the spin chain with adiabatic phonons (1): the main difference being that the self-consistent hopping amplitudes depend functionally on the density (n_j's, or equivalently $s_{z,j}$'s), rather than on spin-spin correlations ($\langle s_i^\dagger s_{i+1} + s_{i+1}^\dagger s_i\rangle$.

ULTRACOLD GASES IN “ARTIFICIAL” NON-ABELIAN FIELDS

Ultracod atoms and HEP. In the recent years the links and interconnections between physics of ultracold atoms and condensed matter physics became solid and well established. Through these links, modern atomic physics reaches the frontiers of the quantum field theory, and has indirect relations to the modern high energy physics. Very recently, it has appeared several proposals on how to use ultracold atoms to simulate abelian $U(1)$ lattice gauge theory [24]. Another research line investigates the possibility of inserting atoms in artificial non-Abelian magnetic fields by employing electromagnetically induced transparence [25], or optical lattices [26]. The recent progress in the latter case will be presented here.

Artificial Abelian "magnetic fields" in a lattice. As it is well known, rapidly rotating harmonically trapped gases of neutral atoms exhibit effects analogous to charged particles in uniform magnetic fields (for a recent overview, see [27]). Thus one should be able to realize analogues of fractional quantum Hall effect (FQHE) in such systems. However, to achieve a regime of “strong” magnetic fields it is desirable to realize artificial fields in an optical lattice by controlling e.g. the phases of the hopping amplitudes. In particular, single particle stationary Schrödinger equation in 2D square lattice in perpendicular magnetic field, in Landau gauge (i.e. with the vector potential $\mathbf{A} = (0, Bx, 0)$) reads:

$$\begin{aligned} E\Psi(m,n) &= -J\left[\Psi(m+1,n) + \Psi(m-1,n) + \exp(-i2\pi\alpha m)\Psi(m,n+1)\right. \\ &+ \left.\exp(+i2\pi\alpha m)\Psi(m,n-1)\right], \end{aligned} \tag{10}$$

where (m,n) are integers enumerating coordinates on the lattice, and $\alpha = Ba^2/2\pi(\hbar c/e)$, is the magnetic flux thought an elementary plaquette of area a^2 in units of quantum flux. Assuming plane waves in the y direction, $\Psi(m,n) = \exp(i\nu n)g(m)$, we obtain the famous Harper’s equation:

$$\varepsilon g(m) = -\left[g(m+1) + g(m-1) + 2\cos(2\pi\alpha m - \nu)g(m)\right]. \tag{11}$$

Jaksch and Zoller [28] were the first to propose how to realize the model of Eq. (10) employing internal states of atoms, and using appropriate combination of laser assisted tunneling, lattice tilting, resonance pumping, etc. They argued that such systems could be used to study the Hofstadter butterfly (i.e. spectrum of states) in the presence of weak atom-atom interactions. Several other groups [29, 30] proposed since then alternative methods of realizing Abelian fields, aiming at the possibility of realizing FQHE in such systems: Laughlin states at low α [30], and analogues of bilayer FQHE (cf. [31]) Halperin states [32].

Artificial non-Abelian "magnetic fields" in a lattice. In [26] we demonstrated that using atoms with multiple internal states ("colours" or "lavours"), one can realize Non-Abelian magnetic fields in an lattice using a similar scheme as [28], i.e. employing laser assisted tunneling, lattice tilting, resonance pumping etc. Generally speaking, the proposed scheme allows, among other things, to generate fields, whose vector potential is linear in x and y, i.e. is of the form $\mathbf{A} = (\hat{M}_x + \hat{N}_x(x/a) + \hat{O}_x(y/a), (\hat{M}_y +$

$\hat{N}_y(x/a)+\hat{O}_y(y/a),0)$, where $\hat{M},\hat{N}$, and $\hat{O}$ are arbitrary hermitian matrices from the algebra of the considered group, which can be chosen practically at will: $SU(n)$, $U(n)$, $GL(n)$ etc. For instance, with $\mathbf{A}=(\hat{M}_x,\hat{N}_y(x/a),0)$ with $\hat{M}_x,N_y$ - hermitian, and $N_y=2\pi\mathrm{diag}(\alpha_1,\alpha_2,\ldots,\alpha_N)$, we obtain the $U(n)$ generalization of the Harper's equation for the n-component wave function:

$$\varepsilon g(m)=-\left[\exp(i\hat{M}_x)g(m+1)+\exp(-iM_x)g(m-1)+2\cos(2\pi\hat{N}_y m-\nu)g(m)\right]. \quad (12)$$

In Ref. [26] we have studied the spectrum of this equation for the $SU(2)$ case, and rational α_1 and α_2. Bloch theory applies in such a case and the spectrum consists of allowed bands and gaps, similarly as in the Hofstadter problem for $n=1$ and $\hat{M}_x=0$. In the latter case the allowed energy band plotted against (rational) α, form the complex fractal figure known as Hofstadter butterfly. In the present case, we plot instead energy gaps. They form a fractal arrangements of genuine holes is the spectrum (Fig. 3), which we term Hostadter moth.

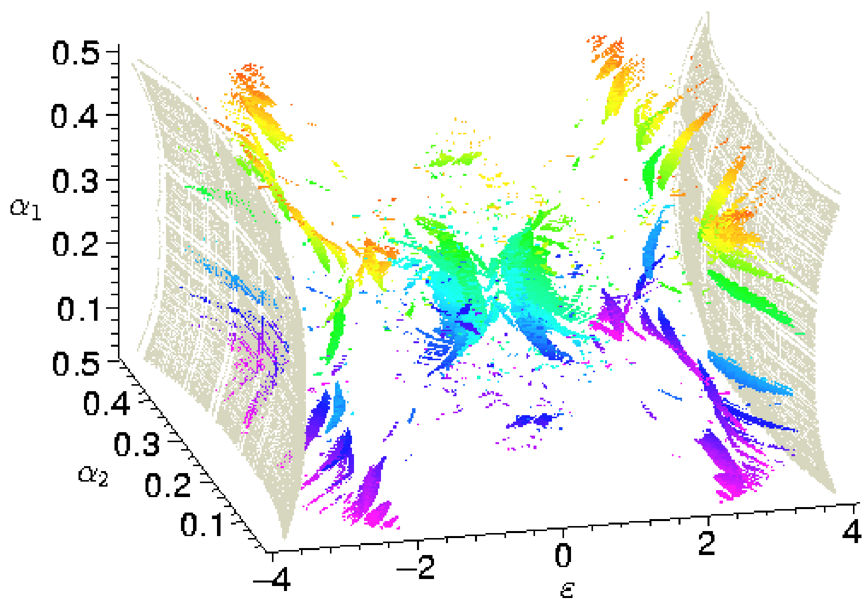

Figure 3. Hofstadter moth's energy gaps plotted in various colours against rational values of "magnetic fluxes" α_1 and α_2. Whole spectrum is contained between the two external, slightly curved, grey "walls".

Despite its richness, Fig. 3 contain relatively little informations on density of states and corresponds to an infinite system. We consider realistic systems of 1000 sites size, and plot a section of the moth for a fixed value of, say, $\alpha_2=2/5$. Such a plot in which the red points (holes) indicate the presence (absence) of an energy eigenvalue is, perhaps, the simplest representation of the density of states (see Fig. 4). The moth section is especially impressive if one turns the figure 90 degrees clockwise. The moth looks like a horrible monster from a horrible Japanese cartoons, that are being shown to the kids all over the world in order to educate them in the spirit of war, killings, anger, etc.

Non-Abelian fractional quantum Hall effect. We term by non-Abelian FQHE the analog of the "standard" (if one dares to use such adjective) FQHE, but in the

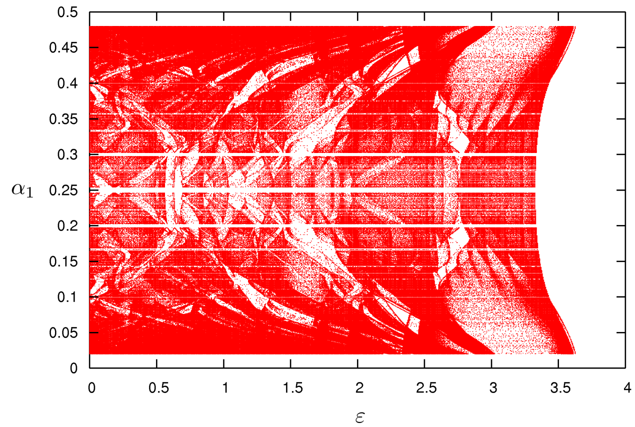

Figure 4. Section of the Hofstadter moth at $\alpha_2 = 2/5$ in a lattice of 1000 sites.

constant non-Abelian magnetic fields. Our starting point is a non-abelian $SU(n)$ field in a 2D square lattice. Note, that we cannot use the gauge potential from the previous subsection, since it leads to a non-zero gauge field tensor. Instead we use $\mathbf{A} = (-B(y-\tilde{b}\hat{S}_y)/2, B(x+\tilde{b}\hat{S}_y)/2, 0)$, where $\hat{S}_{x,y,z}$ are the spin operators ($S = (n-1)/2$). This potential has a non-vanishing constant rotation, and is non-trivially non-Abelian, since its components do not commute. We consider now not too strong fields so that the continuum approximation in the lattice maybe performed. The single particle Hamiltonian becomes thus for $n = 2$

$$H = \frac{(\hat{p}_x + eB(y - b\hat{\sigma}_y)/2c)^2}{2m} + \frac{(\hat{p}_y - eB(x + b\hat{\sigma}_x)/2c)^2}{2m} + \omega\hat{\sigma}_z/2, \tag{13}$$

where we have included σ_z term to allow for non-trivial coupling of different Landau levels. Introducing the operators that annihilate or create Landau level number, $\hat{D} = (\hat{Q} + i\hat{P})\sqrt{2}$, $D^\dagger$, with $\hat{Q} = \sqrt{c/eB}(\hat{p}_x + eBy)$, $\hat{P} = \sqrt{c/eB}(\hat{p}_y - eBx)$, the Hamiltonian attains the form of the Jaynes-Cummings model [21]:

$$H/\hbar\Omega = \left[\hat{D}^\dagger\hat{D} - i\beta(\hat{\sigma}^\dagger\hat{D} - \hat{D}^\dagger\hat{\sigma})\right] + \tilde{\omega}\hat{\sigma}^\dagger\hat{\sigma}, \tag{14}$$

where $\Omega = eB/\hbar mc$, $\tilde{\omega} = \omega/\Omega$, and $\beta = b\sqrt{eB/2c}$. The many body Hamiltonian includes the sum of single particle terms plus the sum of pair interactions of contact type,

$$\mathscr{H} = \sum_i \left[\hat{D}_i^\dagger\hat{D}_i - i\beta(\hat{\sigma}_i^\dagger\hat{D}_i - \hat{D}_i^\dagger\hat{\sigma}_i)\right] + \tilde{\omega}\hat{\sigma}_i^\dagger\hat{\sigma}_i + \frac{g}{2}\sum_{i\neq j}\delta^{(2)}(z_i - z_j), \tag{15}$$

where $z_i = x_i + iy_i$ are complex coordinates in the plane.

There are some similarities between (15) and a multicomponent quantum Hall system studied in the literature (see article by S.M. Girvin and A.H. Monald in [31]), such as electronic systems with spin, or bilayer systems. In the spin analogy our model introduces several novel aspects: i) we may use it for fermions or bosons, ii) we may

consider arbitrary spin (pseudo-spin), and in particular consider contact interactions of the spinor type (cf. [2]). Concerning multilayer analogy we stress that, similarly, we may consider more general "multilayer"=multicolour situations, and control coherent coupling between the "layers". Our model breaks the spin up-down symmetry, and necessarily mixes various Landau levels, if we allow for spin-up excitations. If all spins are down, the ground state (GS) takes the Lauglin form:

$$|\Omega, E=0\rangle = \prod_{i<j}(z_i - z_j)^m \exp(-\sum_i |z_i|^2/4)|\downarrow,\downarrow,\ldots,\downarrow\rangle, \tag{16}$$

where $1/m = \nu$ is the filling factor, $\hat{E} = \sum_i \hat{D}_i^\dagger \hat{D}_i$ is the total number of excitations, and E its mean value. Correspondingly, the GS with one excitation will be

$$|\Omega, |E=1\rangle\rangle = \prod_{i<j}(z_i - z_j)^m \exp(-\sum_i |z_i|^2/4)|W(\uparrow)\rangle| + \sum_i D_i^\dagger \Omega, E=1\rangle \tag{17}$$

It is natural to generalize the concept of hole excitations for the non-Abelian FQHE, defining single hole states $\prod_i (z_i - Z_a)|\omega, E\rangle$, two hole states $\prod_i (z_i - Z_a)(z_i - Z_b)|\omega, E\rangle$ etc., where Z_a, Z_b are $n \times n$ matrices. These excitations constitute obviously fractional anyons, but currently we are working on checking whether they themselves are non-Abelian, i.e. transform according the non-Abelian representations of the permutation group [33]). This has been a long lasting quest for non-Abelian anyons in the condensed matter literature, but without no clear experimental observation so far. The most prominent candidate is electronic (fermionic) $\nu = 5/2$ FQHE state. This state has been observed in experiments, Moore and Read [34], and independently Greiter, Wen and Wilczek [35] proposed to explain it in terms of the "Pfaffian state" (see also [31]). Recently, however, Töke and Jain [36] proposed an alternative "composite fermions" model which does not relate to non-Abelian statistics in any obvious manner. For bosons, a promising candidate is the $\nu = 3/2$ state from the so called Read-Rezayi sequence of incompressible correlated liquids. This state seem to be a true ground state of the rapidly rotating gas of bosons interacting via contact (Van der Waals) forces with a moderate amount of dipolar interactions [37]. Such situation may be achieved, for instance, with Bose condensed Chromium, as in experiments of the T. Pfau's group [38]. We hope that non-Abelian FQHE, due its profound and direct non-commutative character, will provide further, experimentally feasible examples of non-Abelian anyons.

We conclude with the standard conclusion of M. Lewenstein and the motto to the review [2], which is a citation from William Shakespeare's "Hamlet": *There are more things in heaven and earth, Horatio, than are dreamed of in your philosophy*. In the present context, it expresses our neverending curiosity, enthusiasm, joy, and excitement of working in atomic physics in general, and physics the of ultracold atoms in particular.

We thank B. Damski, E. Demler and H. Ritsch for discussions. We acknowledge support from German Government(SFB 407, SPP 1116, GK 282, 436 POL), Spanish Government (FIS2005-04627,01369). EU IP Programme "SCALA", ESF PESC QUDEDIS. G.M, C. M and J.L are supported by Spanish Ramon y Cajal Program, EC-Marie Curie fellowship and Swedish Goverment/Vetenkasprader respectively.

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